4 October 2001
PHYSICS LETTERS B
ELSEVIER
Physics Letters B 517 (2001) 450–456
www.elsevier.com/locate/npe

Testing non-commutative QED, constructing non-commutative MHD

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Received 7 June 2001; accepted 1 August 2001
Editor: L. Alvarez-Gaumé

Abstract

The effect of non-commutativity on electromagnetic waves violates Lorentz invariance: in the presence of a background magnetic induction field **b**, the velocity for propagation transverse to **b** differs from c, while propagation along **b** is unchanged. In principle, this allows a test by the Michelson–Morley interference method. We also study non-commutativity in another context, by constructing the theory describing a charged fluid in a strong magnetic field, which forces the fluid particles into their lowest Landau level, and renders the fluid dynamics non-commutative, with a Moyal product determined by the background magnetic field.

1. Introduction

The idea that spatial coordinates do not commutate [1] has a well-known realization in physics: the quantized motion of particles in a magnetic field, sufficiently strong so that projection on the lowest Landau level can be justified, is described by non-commuting coordinates on the plane perpendicular to the field [2]. Recently this phenomenon has played a role in various quantum mechanical studies, involving both theoretical models [3] and phenomenological applications [4]. At the same time generalizations to quantum field theory have also been made, giving rise to various "non-commutative" field theories, for example, non-commutative quantum electrodynamics.

In Section 2 of this Letter we examine the effect of an external magnetic field on non-commutative photon dynamics, i.e., electrodynamics without charged particles, which nevertheless is a non-linear theory owing to its non-commutativity. We show that the velocity of light depends on the direction of propagation relative to the external magnetic field, thus allowing for a Michelson–Morley-type test of non-commutativity, which evidently violates special relativity (Lorentz invariance).

In Section 3 we study a magnetohydrodynamical (MHD) field theory in an intense magnetic field, which effects a field theoretical analog for the previously mentioned reduction to the lowest Landau level, and results in non-commutative MHD, in complete analogy to what happens to particles in a strong magnetic field [2]. Here the non-commutativity manifests itself

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PII: S0370-2693(01)00986-8

in the charged fluid density not commuting with itself. The form of the non-commutativity depends on the nature of the density: if the fluid is structureless, with quantum commutators deduced from Poisson brackets, one particular expression is obtained. When the fluid is constructed from point particles, whose coordinates do not commute, then the density commutator involves a Moyal phase, which reduces to the previous expression in a semi-classical limit. Relevant formulas are expressed succinctly with the help of the "star" product.

2. Testing non-commutative QED

The non-commutative generalization for the free Maxwell–Lagrange density involves the "star" product of the non-commutative field strength $\widehat{F}_{\mu\nu}$, constructed from the potential $\hat{A}_\mu$,

$$\widehat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - ig(\hat{A}_\mu * \hat{A}_\nu - \hat{A}_\nu * \hat{A}_\mu), \tag{1}$$

$$\widehat{\mathcal{L}} = -\frac{1}{4}\widehat{F}_{\mu\nu} * \widehat{F}^{\mu\nu}, \tag{2}$$

where the star product is defined by

$$(f * g)(x) = e^{\frac{i}{2}\theta^{\alpha\beta}\partial_\alpha \partial'_\beta} f(x) g(x')\big|_{x'=x}. \tag{3}$$

The non-linear terms in (1) enter with the coupling $g = \frac{e}{\hbar c}$. To first order in $\theta^{\alpha\beta} = -\theta^{\beta\alpha}$, $\widehat{\mathcal{L}}$ may be expressed in terms of the conventional Maxwell tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{4}$$

with $\hat{A}_\mu$ related to A_μ by

$$\hat{A}_\mu = A_\mu - \frac{1}{2}\theta^{\alpha\beta} A_\alpha (\partial_\beta A_\mu + F_{\beta\mu}), \tag{5}$$

$$\widehat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \theta^{\alpha\beta} A_\alpha \partial_\beta F_{\mu\nu}, \tag{6}$$

with g absorbed in θ. It follows that apart from a total derivative term, which does not affect the equations of motion [5],

$$\begin{aligned}\widehat{\mathcal{L}} = {} & -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{8}\theta^{\alpha\beta}F_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} \\ & - \frac{1}{2}\theta^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}F^{\mu\nu} + \mathcal{O}(\theta^2).\end{aligned} \tag{7}$$

Our strategy is to solve the equations of motion implied by (7) (to first order in θ) and to exhibit how special relativity is violated. Henceforth we take $\theta^{\alpha\beta}$ to have only spatial components, $\theta^{0\alpha} = 0$, $\theta^{ij} = \epsilon^{ijk}\theta^k$, and work exclusively with the field strengths $F^{i0} = E^i$ and $F_{ij} = -\epsilon_{ijk}B^k$, rather than with the vector potential.

The "Maxwell" equations that follow from (7) are

$$\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} + \nabla \times \mathbf{E} = 0, \tag{8}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{9}$$

These of course are a consequence of (4). The other equations reflect the non-linear dynamics of (7), and can be written in terms of a displacement field $\mathbf{D}$ and magnetic field $\mathbf{H}$

$$\frac{1}{c}\frac{\partial}{\partial t}\mathbf{D} - \nabla \times \mathbf{H} = 0, \tag{10}$$

$$\nabla \cdot \mathbf{D} = 0. \tag{11}$$

Constitutive relations follow from (7)

$$\mathbf{D} = (1 - \boldsymbol{\theta} \cdot \mathbf{B})\mathbf{E} + (\boldsymbol{\theta} \cdot \mathbf{E})\mathbf{B} + (\mathbf{E} \cdot \mathbf{B})\boldsymbol{\theta}, \tag{12}$$

$$\mathbf{H} = (1 - \boldsymbol{\theta} \cdot \mathbf{B})\mathbf{B} + \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)\boldsymbol{\theta} - (\boldsymbol{\theta} \cdot \mathbf{E})\mathbf{E}. \tag{13}$$

We seek solutions to (8)–(13) where the electric field is a propagating plane wave

$$\mathbf{E} = \mathbf{E}(\omega t - \mathbf{k} \cdot \mathbf{r}). \tag{14}$$

Eq. (8) implies that

$$\mathbf{B} = \boldsymbol{\kappa} \times \mathbf{E} + \mathbf{b}, \tag{15}$$

where $\boldsymbol{\kappa} = c\mathbf{k}/\omega$ and $\mathbf{b}$ is a time-independent background magnetic induction field, which must be transverse according to (9). However, we shall specialize by taking $\mathbf{b}$ to be constant.

From (12)–(15) it follows that $\mathbf{D}$ and $\mathbf{H}$ are functions of $\omega t - \mathbf{k} \cdot \mathbf{r}$, so (10) implies that

$$\mathbf{D} = -\boldsymbol{\kappa} \times \mathbf{H} + \mathbf{d}, \tag{16}$$

where again $\mathbf{d}$ is a time-independent transverse background. We assume that no background field contributes to $\mathbf{D}$, so $\mathbf{d}$ is chosen to cancel the constant contribution to $-\boldsymbol{\kappa} \times \mathbf{H}$ coming from $\mathbf{b}$.

After $\mathbf{D}$ and $\mathbf{H}$ are expressed in terms of $\mathbf{E}$ using (12), (13) and (15), Eq. (16) becomes

$$\begin{aligned}&(1 - \boldsymbol{\theta} \cdot \mathbf{b})E^i + \epsilon^{ijk}\kappa^j E_T^k (\mathbf{E} \cdot \boldsymbol{\theta}) \\ &\quad - E^i \epsilon^{jkl}\theta^j \kappa^k E_T^l + \beta^{ij} E^j \\ &\qquad = \kappa^2 E_T^i - \kappa^2 E_T^i \epsilon^{jkl}\theta^j \kappa^k E_T^l\end{aligned}$$

$$- \frac{1}{2}\epsilon^{ijk}\kappa^j\theta^k E^2\big(1-\kappa^2\big) - \frac{\kappa^2}{2}\epsilon^{ijk}\kappa^j\theta^k E_L^2 + \epsilon^{ijk}\kappa^j E_T^k(\mathbf{E}\cdot\boldsymbol{\theta}) + \kappa^2\big(\hat\kappa^j\hat\kappa^k\beta^{jk} - \beta^j_j - \boldsymbol{\theta}\cdot\mathbf{b}\big)E_T^i + \kappa^2\beta^{ij}E_T^j - \kappa^i\kappa^j\beta^{jk}E_T^k, \tag{17}$$

where $\hat{\boldsymbol{\kappa}}$ is the unit vector $\boldsymbol{\kappa}/|\boldsymbol{\kappa}|$, and $\beta^{ij} = \theta^i b^j + \theta^j b^i$. The electric field has been decomposed into transverse and longitudinal parts, $\mathbf{E} = \mathbf{E}_T + \hat{\boldsymbol{\kappa}} E_L$, with $\hat{\boldsymbol{\kappa}}\cdot\mathbf{E}_T = 0$. Note that $\epsilon^{ijk}\kappa^j E_T^k(\mathbf{E}\cdot\boldsymbol{\theta})$ cancels from both sides of the equality. By projecting the above on $\hat{\boldsymbol{\kappa}}$, we arrive at an expression for the longitudinal component of $\mathbf{E}$.

$$\big(1-\boldsymbol{\theta}\cdot\mathbf{b} - \epsilon^{jkl}\theta^j\kappa^k E_T^l\big)E_L + \hat\kappa^i\beta^{ij}E^j = 0. \tag{18}$$

To lowest order in θ, this give for E_L

$$E_L = -\hat\kappa^i\beta^{ij}E_T^j. \tag{19}$$

Reinserting this in (17) and keeping terms at most linear in θ leaves

$$\big(1-\kappa^2\big)\Big[\big(1-\boldsymbol{\theta}\cdot\mathbf{b} - \epsilon^{jkl}\theta^j\kappa^k E_T^l\big)E_T^i + \beta^{ij}E_T^j - \hat\kappa^i\hat\kappa^j\beta^{jk}E_T^k + \frac{1}{2}\epsilon^{ijk}\kappa^j\theta^k E_T^2\Big] = \kappa^2\big(\hat\kappa^j\hat\kappa^k\beta^{jk} - \beta^j_j\big)E_T^i. \tag{20}$$

In the absence of non-commutativity ($\boldsymbol{\theta} = 0$) the above reduces to $(1-\kappa^2)E_T^i = 0$, which implies the usual dispersion result $\kappa^2 = 1$ or $\omega^2 = c^2k^2$. The same dispersion law holds when there is no background field $\mathbf{b}$. However, when both $\boldsymbol{\theta}$ and $\mathbf{b}$ are non-vanishing, $\kappa^2 = 1$ is no longer a solution; rather we must take $1-\kappa^2$ to be $\mathcal{O}(\theta)$. Then to lowest, linear order in θ, (19) becomes

$$\big(1-\kappa^2\big)\mathbf{E}_T = \big(\hat\kappa^j\hat\kappa^k\beta^{jk} - \beta^j_j\big)\mathbf{E}_T = -2\boldsymbol{\theta}_T\cdot\mathbf{b}_T\mathbf{E}_T. \tag{21}$$

Thus the solution demands a modified dispersion law:

$$\kappa^2 = \frac{c^2k^2}{\omega^2} = 1 + 2\boldsymbol{\theta}_T\cdot\mathbf{b}_T \tag{22}$$

or

$$\omega = ck(1-\boldsymbol{\theta}_T\cdot\mathbf{b}_T). \tag{23}$$

The same result may be obtained more quickly and easily, if less reliably, by linearizing the constitutive Eqs. (12) and (13) around the background magnetic induction field $\mathbf{b}$. Then (12) and (13) read

$$D^i = \varepsilon^{ij}E^j, \tag{24}$$

$$H^i = \big(\mu^{-1}\big)^{ij}B^j, \tag{25}$$

where the electric permitivity is given by

$$\varepsilon^{ij} = \delta^{ij}(1-\boldsymbol{\theta}\cdot\mathbf{b}) + \beta^{ij} \tag{26}$$

and the inverse magnetic permeability by

$$\big(\mu^{-1}\big)^{ij} = \delta^{ij}(1-\boldsymbol{\theta}\cdot\mathbf{b}) - \beta^{ij}. \tag{27}$$

It is now posited that all dynamical fields are functions of $\omega t - \mathbf{k}\cdot\mathbf{r}$, and that $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{H}$ have no background field contributions. With

$$\mathbf{B} = \boldsymbol{\kappa}\times\mathbf{E} \tag{28}$$

and

$$\mathbf{D} = -\boldsymbol{\kappa}\times\mathbf{H}, \tag{29}$$

it follows from (24) and (25) that

$$D^i = -\epsilon^{ijk}\kappa^j\big(\mu^{-1}\big)^{kl}\epsilon^{lmn}\kappa^m\big(\varepsilon^{-1}\big)^{nq}D^q. \tag{30}$$

To first order in θ

$$\big(\varepsilon^{-1}\big)^{ij} = \delta^{ij}(1+\boldsymbol{\theta}\cdot\mathbf{b}) - \beta^{ij}. \tag{31}$$

Inserting this and (27) into (30) gives

$$D^i = \kappa^2\big(1 + \hat\kappa^j\hat\kappa^k\beta^{jk} - \beta^j_j\big)D^i, \tag{32}$$

whose solution is again (23).

To recapitulate, we see that a plane electromagnetic wave does not see the non-commutativity if the background magnetic induction field $\mathbf{b}$ vanishes, or if the wave propagates in the direction of $\mathbf{b}$. On the other hand, propagation transverse to $\mathbf{b}$ is at a velocity that differs from c by the factor $1-\boldsymbol{\theta}_T\cdot\mathbf{b}_T$. Note that both polarizations travel at the same (modified) velocity, so there is no Faraday-like rotation. Let us also observe that the effective Lagrange density (7) possesses two interaction terms proportional to θ, with definite numerical constants. Owing to the freedom of rescaling θ, only their ratio is significant. It is straightforward to verify that if the ratio is different from what is written in (7), the two linear polarizations travel at different velocities. Thus the non-commutative theory is unique in affecting the two polarizations equally, at least to $\mathcal{O}(\theta)$ [6].

The change in velocity for motion relative to an external magnetic induction **b** allows searching for the effect with a Michelson–Morley experiment. In a conventional apparatus with two legs of length ℓ_1 and ℓ_2 at right angles to each other, a light beam of wavelength λ is split in two, and one ray travels along **b** (where there is no effect), while the other, perpendicular to **b**, feels the change of velocity and interferes with the the first. After rotating the apparatus by 90°, the interference pattern will shift by $2(\ell_1+\ell_2)\boldsymbol{\theta}_T\cdot\mathbf{b}_T/\lambda$ fringes. Taking light in the visible range, $\lambda\sim 10^{-5}$ cm, a field strength $b\sim 1$ tesla, and using the current bound on $\theta\leqslant(10\ \text{TeV})^{-2}$ obtained in [7], one finds that a length $\ell_1+\ell_2\geqslant 10^{18}$ cm ~ 1 parsec would be required for a shift of one fringe. Galactic magnetic fields are neither that strong nor coherent over such large distances, so another experimental setting needs to found to test for non-commutativity.

Finally we note that there is close connection between our results on photon propagation and the general analysis of Lorentz non-invariant modifications to the standard model [8].

3. Constructing non-commutative MHD

3.1. *Particle non-commutativity in the lowest Landau level*

In order to describe the motion of a charged fluid in an intense magnetic field, which effects a reduction to the field-theoretical analog of the lowest Landau level and results in a non-commutative field theory, we review the story for point particles on a plane, with an external magnetic field **b** perpendicular to the plane [2]. The equation for the 2-vector $\mathbf{r}=(x,y)$ is

$$m\dot{v}^i=\frac{e}{c}\epsilon^{ij}v^jb+f^i(\mathbf{r}),\tag{33}$$

where **v** is the velocity $\dot{\mathbf{r}}$, and **f** represents other forces, which we take to be derived from a potential V: $\mathbf{f}=-\nabla V$. The limit of large b is equivalent to small m. Setting the mass to zero in (33) leaves a first order equation.

$$\dot{r}^i=\frac{c}{eb}\epsilon^{ij}f^j(\mathbf{r}).\tag{34}$$

This may be obtained by taking Poisson brackets of **r** with the Hamiltonian

$$H_0=V\tag{35}$$

provided the fundamental brackets describe non-commutative coordinates,

$$\{r^i,r^j\}=\frac{c}{eb}\epsilon^{ij}\tag{36}$$

so that

$$\dot{r}^i=\{H_0,r^i\}=\{r^j,r^i\}\partial_jV=\frac{c}{eb}\epsilon^{ij}f^j(\mathbf{r}).\tag{37}$$

The non-commutative algebra (36) and the associated dynamics can be derived in the following manner. The Lagrangian for the equation of motion (33) is

$$L=\frac{1}{2}mv^2+\frac{e}{c}\mathbf{v}\cdot\mathbf{A}-V,\tag{38}$$

where we choose the gauge $\mathbf{A}=(0,bx)$. Setting m to zero leaves

$$L_0=\frac{eb}{c}x\dot{y}-V(x,y),\tag{39}$$

which is of the form $p\dot{q}-h(p,q)$, and one sees that $(\frac{eb}{c}x,y)$ form a canonical pair. This implies (36), and identifies V as the Hamiltonian.

Finally, we give a canonical derivation of non-commutativity in the $m\to 0$ limit, starting with the Hamiltonian

$$H=\frac{\boldsymbol{\pi}^2}{2m}+V.\tag{40}$$

H gives (33) upon bracketing with **r**, provided the following brackets hold;

$$\{r^i,r^j\}=0,\tag{41}$$

$$\{r^i,\pi^j\}=\delta^{ij},\tag{42}$$

$$\{\pi^i,\pi^j\}=-\frac{eb}{c}\epsilon^{ij}.\tag{43}$$

Here $\boldsymbol{\pi}$ is the kinematical (non-canonical) momentum, $m\dot{\mathbf{r}}$, related to the canonical momentum **p** by $\boldsymbol{\pi}=\mathbf{p}-\frac{e}{c}\mathbf{A}$.

We wish to set m to zero in (40). This can only be done provided $\boldsymbol{\pi}$ vanishes, and we impose $\boldsymbol{\pi}=0$ as a constraint. But according to (43), the bracket of the constraints $C^{ij}=-\frac{eb}{c}\epsilon^{ij}$ is non-zero. Hence we must introduce Dirac brackets:

$$\{O_1,O_2\}_D=\{O_1,O_2\}-\{O_1,\pi^k\}\big(C^{-1}\big)^{kl}\{\pi^l,O_2\}.\tag{44}$$

With (44), any Dirac bracket involving $\boldsymbol{\pi}$ vanishes, so $\boldsymbol{\pi}$ may indeed be set to zero. But the Dirac bracket of two coordinates is now non-vanishing.

$$\{r^i, r^j\}_D = -\{r^i, \pi^k\}\frac{c}{eb}\epsilon^{kl}\{\pi^l, r^j\} = \frac{c}{eb}\epsilon^{ij}. \tag{45}$$

In this approach, non-commuting coordinates arise as Dirac brackets in a system constrained to lie in the lowest Landau level.

3.2. *Field non-commutativity in the lowest Landau level*

We now turn to the equations of a charged fluid with density ρ and mass parameter m (introduced for dimensional reasons) moving on a plane with velocity $\mathbf{v}$ in an external magnetic field perpendicular to the plane. ρ and $\mathbf{v}$ are functions of t and $\mathbf{r}$ and give an Eulerian description of the fluid. The equations that are satisfied are the continuity equation

$$\dot\rho + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{46}$$

and the Euler equation.

$$m\dot v^i + m\mathbf{v}\cdot\nabla v^i = \frac{e}{c}\epsilon^{ij}v^j b + f^i. \tag{47}$$

Here f^i describes additional forces, e.g., $-\frac{1}{\rho}\nabla P$ where P is pressure. We shall take the force to be derived from a potential of the form

$$\mathbf{f}(\mathbf{r}) = -\nabla\frac{\delta}{\delta\rho(\mathbf{r})}\int d^2r\, V. \tag{48}$$

[For isentropic systems, the pressure is only a function of ρ; (48) holds with V a function of ρ, related to the pressure by $P(\rho) = \rho V'(\rho) - V(\rho)$. Here we allow more general dependence of V on ρ (e.g., nonlocality or dependence on derivatives of ρ) and also translation non-invariant, explicit dependence on $\mathbf{r}$.]

Eqs. (46) and (47) follow by bracketing ρ and $\mathbf{v}$ with the Hamiltonian

$$H = \int d^2r\left(\rho\frac{\boldsymbol{\pi}^2}{2m} + V\right) \tag{49}$$

provided that fundamental brackets are taken as

$$\{\rho(\mathbf{r}), \rho(\mathbf{r}')\} = 0, \tag{50}$$

$$\{\pi(\mathbf{r}), \rho(\mathbf{r}')\} = \nabla\delta(\mathbf{r}-\mathbf{r}'), \tag{51}$$

$$\{\pi^i(\mathbf{r}), \pi^j(\mathbf{r}')\} = -\epsilon^{ij}\frac{1}{\rho}\left(m\omega(\mathbf{r}) + \frac{eb}{c}\right)\delta(\mathbf{r}-\mathbf{r}'), \tag{52}$$

where $\epsilon^{ij}\omega(\mathbf{r})$ is the vorticity $\partial_i v^j - \partial_j v^i$, and $\boldsymbol{\pi} = m\mathbf{v}$ [9].

We now consider a strong magnetic field and take the limit $m\to 0$, which is equivalent to large b. Eqs. (47) and (48) reduce to

$$v^i = -\frac{c}{eb}\epsilon^{ij}\frac{\partial}{\partial r^j}\frac{\delta}{\delta\rho(\mathbf{r})}\int d^2r\, V. \tag{53}$$

Combining this with the continuity equation (46) gives the equation for the density "in the lowest Landau level".

$$\dot\rho(\mathbf{r}) = \frac{c}{eb}\frac{\partial}{\partial r^i}\rho(\mathbf{r})\epsilon^{ij}\frac{\partial}{\partial r^j}\frac{\delta}{\delta\rho(\mathbf{r})}\int d^2r\, V. \tag{54}$$

(For the right-hand side not to vanish, V must not be solely a function of ρ.)

The equation of motion (54) can be obtained by bracketing with the Hamiltonian

$$H_0 = \int d^2r\, V \tag{55}$$

provided the charge density bracket is non-vanishing, showing non-commutativity of the ρ's.

$$\{\rho(\mathbf{r}), \rho(\mathbf{r}')\} = -\frac{c}{eb}\epsilon^{ij}\partial_i\rho(\mathbf{r})\partial_j\delta(\mathbf{r}-\mathbf{r}'). \tag{56}$$

H_0 and this bracket may be obtained from (49) and (50)–(52) with the same Dirac procedure presented for the particle case: We wish to set m to zero in (49); this is possible only if $\boldsymbol{\pi}$ is constrained to vanish. But the bracket of the $\boldsymbol{\pi}$'s is non-vanishing, even at $m = 0$, because $b \neq 0$. Thus at $m = 0$ we posit the Dirac brackets

$$\begin{aligned}\{O_1(\mathbf{r}_1), O_2(\mathbf{r}_2)\}_D &= \{O_1(\mathbf{r}_1), O_2(\mathbf{r}_2)\} \\ &\quad - \int d^2\mathbf{r}'_1\, d^2\mathbf{r}'_2\,\{O_1(\mathbf{r}_1), \pi^i(\mathbf{r}'_1)\} \\ &\quad \times (C^{-1})^{ij}(\mathbf{r}'_1, \mathbf{r}'_2)\{\pi^j(\mathbf{r}'_2), O_2(\mathbf{r}_2)\},\end{aligned} \tag{57}$$

where

$$(C^{-1})^{ij}(\mathbf{r}_1, \mathbf{r}_2) = \frac{c}{eb}\epsilon^{ij}\rho(\mathbf{r}_1)\delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{58}$$

Hence Dirac brackets with $\boldsymbol{\pi}$ vanish, and the Dirac bracket of densities is non-vanishing as in (56).

$$\begin{aligned}\{\rho(\mathbf{r}), \rho(\mathbf{r}')\}_D &= -\frac{c}{eb}\int d^2r''\,\{\rho(\mathbf{r}), \pi^i(\mathbf{r}'')\}\rho(\mathbf{r}'') \\ &\quad \times \epsilon^{ij}\{\pi^j(\mathbf{r}''), \rho(\mathbf{r}')\} \\ &= -\frac{c}{eb}\epsilon^{ij}\partial_i\rho(\mathbf{r})\partial_j\delta(\mathbf{r}-\mathbf{r}').\end{aligned} \tag{59}$$

The ρ-bracket enjoys a more appealing expression in momentum space. Upon defining

$$\tilde{\rho}(\mathbf{p}) = \int d^2r\, e^{i\mathbf{p}\cdot\mathbf{r}} \rho(\mathbf{r}) \tag{60}$$

we find

$$\{\tilde{\rho}(\mathbf{p}), \tilde{\rho}(\mathbf{q})\}_D = -\frac{c}{eb}\epsilon^{ij} p^i q^j \tilde{\rho}(\mathbf{p}+\mathbf{q}). \tag{61}$$

The brackets (56), (61) give the algebra of area preserving diffeomorphisms [10].

A Lagrangian derivation, analogous to the particle case (38), (39) is problematic and not available. The difficulty is that the Poisson structures (50)–(52) and (56), (59) are irregular: there exist "Casimirs" whose brackets with the dynamical variables vanish. For (50)–(52) the Casimirs comprise the tower

$$C^n = \int d^2r\, \rho^{1-n} \left(m\omega + \frac{eb}{c} \right)^n$$

with n arbitrary; while for (56), (59) they read

$$C_0^n = \int d^2r\, \rho^n,$$

again with arbitrary n. (Evidently C_0^n is equivalent to the $m = 0$ limit of C^n.) Consequently the Poisson structures do not poses an inverse; no symplectic 2-form can be found in terms of the above variables, and no canonical 1-form can be added to the Hamiltonian for a construction of a Lagrangian. (By introducing different, redundant variables one can remove this obstacle, at least in the finite m case [11].)

The form of the charge density bracket (56), (59), (61) can be understood by reference to the particle substructure for the fluid. Take

$$\rho(\mathbf{r}) = \sum_n \delta(\mathbf{r} - \mathbf{r}_n), \tag{62}$$

where n labels the individual particles. The coordinates of each particle satisfy the non-vanishing bracket (36). Then the $\{\rho(\mathbf{r}), \rho(\mathbf{r}')\}$ bracket takes the form (56), (59), (61).

3.3. Quantization of non-commutative MHD

Quantization before the reduction to the lowest Landau level is straightforward. For the particle case (41)–(43) and for the fluid case (50)–(52) we replace brackets with $i/\hbar$ times commutators. After reduction to the lowest Landau level we do the same for the particle case thereby arriving at the "Peierls substitution", which states that the effect of an impurity [V in (38)] on the lowest Landau energy level can be evaluated to lowest order by viewing the (x, y) arguments of V as non-commuting variables [2].

However, for the fluid case quantization presents a choice. On the one hand, we can simply promote the bracket (56), (59), (61) to a commutator by multiplying by $i/\hbar$.

$$[\rho(\mathbf{r}), \rho(\mathbf{r}')] = i\hbar\frac{c}{eb}\epsilon^{ij}\partial_i\rho(\mathbf{r}')\partial_j\delta(\mathbf{r}-\mathbf{r}'), \tag{63}$$

$$[\tilde{\rho}(\mathbf{p}), \tilde{\rho}(\mathbf{q})] = i\hbar\frac{c}{eb}\epsilon^{ij} p^i q^j \tilde{\rho}(\mathbf{p}+\mathbf{q}). \tag{64}$$

Alternatively we can adopt the expression (62), for the operator $\rho(\mathbf{r})$, where the $\mathbf{r}_n$ now satisfy the non-commutative algebra

$$[r_n^i, r_{n'}^j] = -i\hbar\frac{c}{eb}\epsilon^{ij}\delta_{nn'} \tag{65}$$

and calculate the ρ commutator as a derived quantity.

However, once $\mathbf{r}_n$ is a non-commuting operator, functions of $\mathbf{r}_n$, even δ-functions, have to be ordered. We choose the Weyl ordering, which is equivalent to defining the Fourier transform as

$$\tilde{\rho}(\mathbf{p}) = \sum_n e^{i\mathbf{p}\cdot\mathbf{r}_n}. \tag{66}$$

With the help of (65) and the Baker–Hausdorff lemma, we arrive at the "trigonometric algebra" [12]

$$[\tilde{\rho}(\mathbf{p}), \tilde{\rho}(\mathbf{q})] = 2i \sin\left(\frac{\hbar c}{2eb}\epsilon^{ij} p^i q^j\right)\tilde{\rho}(\mathbf{p}+\mathbf{q}). \tag{67}$$

This reduces to (64) for small $\hbar$.

This form for the commutator, (67), is connected to a Moyal star product [13] in the following fashion. For an arbitrary c-number function $f(\mathbf{r})$ define

$$\langle f \rangle = \int d^2r\, \rho(\mathbf{r}) f(\mathbf{r}) = \frac{1}{(2\pi)^2}\int d^2p\, \tilde{\rho}(\mathbf{p})\tilde{f}(-\mathbf{p}). \tag{68}$$

Multiplying (67) by $\tilde{f}(-\mathbf{p})\tilde{g}(-\mathbf{q})$ and integrating gives

$$[\langle f \rangle, \langle g \rangle] = \langle h \rangle, \tag{69}$$

with

$$h(\mathbf{r}) = (f * g)(\mathbf{r}) - (g * f)(\mathbf{r}), \tag{70}$$

where the “$*$” product is defined as

$$(f * g)(\mathbf{r}) = e^{\frac{i}{2}\frac{\hbar c}{eb}\epsilon^{ij}\partial_i\partial'_j} f(\mathbf{r})g(\mathbf{r}')\big|_{\mathbf{r}'=\mathbf{r}}. \tag{71}$$

Note however that only the commutator is mapped into the star commutator. The product $\langle f\rangle\langle g\rangle$ is not equal to $\langle f * g\rangle$.

The lack of consilience between (64) and (67) is an instance of the Groenwald–VanHove theorem which establishes the impossibility of taking over into quantum mechanics all classical brackets [13]. Eqs. (67)–(71) explicitly exhibit the physical occurrence of the star product for fields in a strong magnetic background.

Note added

A result identical to ours has been reported by Cai [14].

Acknowledgements

Z.G. thanks W. Skiba for discussions. R.J. acknowledges S. Carroll for information on Ref. [5]. A.P. is grateful to City College, CUNY, for hospitality during part of this work. This work is supported in part by funds provided by the US Department of Energy (D.O.E.) under cooperative research agreements DE-FC02-94ER40818 and DE-FG02-91ER40676.

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